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ON THE ELLIPTIC EQUATIONS $\Delta u = K(x)u^\sigma$ AND $\Delta u = K(x)e^{2u}$

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ABSTRACT. We give some nonexistence results for the equations $\Delta u = K(x)u^\sigma$ and $\Delta u = K(x)e^{2u}$ for $K(x) \geqslant 0$.

1. Introduction. In this paper we study the elliptic equations

$$\Delta u = K(x)u^\sigma \quad \text{in } \mathbf{R}^n \tag{1.1}$$

and

$$\Delta u = K(x)e^{2u} \quad \text{in } \mathbf{R}^n, \tag{1.2}$$

where $\sigma > 1$ is a constant, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ and $K(\cdot)$ is a bounded Hölder continuous function in $\mathbf{R}^n$. We are concerned with the existence problems of locally bounded and positive solutions for (1.1) and locally bounded solutions for (1.2).

These problems come from geometry. We give a brief description and refer the details to Kazdan and Warner [5] and Ni [13, 14]. Let (M, g) be a Riemannian manifold of dimension n, $n \geqslant 2$, and $K(\cdot)$ be a given function on M. We ask the following question: can one find a new metric g_1 on M such that K is the scalar curvature of g_1 and g_1 is conformal to g (i.e., $g_1 = \psi g$ for some function $\psi > 0$ on M)? In the case $n \geqslant 3$, we write $\psi = u^{4/(n-2)}$. Then this problem is equivalent to the problem of finding positive solutions of the equation

$$\frac{4(n-1)}{n-2}\Delta u - ku + Ku^{(n+2)/(n-2)} = 0, \tag{1.3}$$

where Δ, k are the Laplacian and scalar curvature in the g metric, respectively. In the case $M = \mathbf{R}^n$ and $g = (\delta_{ij})$, then $k = 0$ and equation (1.3) reduces to (1.1) with $\sigma = (n+2)/(n-2)$, after an appropriate scaling and sign changing of $K(\cdot)$. In the case $n = 2$, we write $\psi = e^{2u}$. Then this problem is equivalent to the problem of finding locally bounded solutions of the equation

$$\Delta u - k + Ke^{2u} = 0, \tag{1.4}$$

where Δ, k are the Laplacian and Gaussian curvature on M in the g metric. In the case $M = \mathbf{R}^2$ and $g = (\delta_{ij})$, we have $k = 0$ and equation (1.4) reduces to (1.2), after a sign changing of K.

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In [13 and 14], Ni makes major contributions to the existence of solutions for (1.1) and (1.2). After these two papers, there are many improved results published, such as McOwen [10, 11], Naito [12], Kawano, Kusano and Naito [3], Kawano and Kusano [4], Kusano and Oharu [7], Ding and Ni [1], Kusano, Swanson and Usami [8] and Lin [9].

In this paper, we consider the case $K(x) \geqslant 0$ in (1.1) and (1.2). We obtain some nonexistence results which make the understanding of the case $K(x) \geqslant 0$ almost complete. We divide this paper into two parts. In Part I, we consider (1.1). Thus we consider the case (1.1) with $n \geqslant 3$ in §2, (1.1) with $n = 2$ in §3 and (1.1) with $n = 1$ in §4. We consider (1.2) in Part II. Thus we consider the case (1.2) with $n \geqslant 3$ in §5, (1.2) with $n = 2$ in §6 and (1.2) with $n = 1$ in §7.

We remark that the technique of the proof of the main nonexistence theorem is essentially equivalent to the proof of Keller [6]. We thank the referee for bringing the reference [6] to our attention.

PART I. $\Delta u = K(x)u^\sigma$

2. The case $n \geqslant 3$. In this case, Ni [13] proves the main existence result: Let K be bounded. If $|K(x)| \leqslant C/|x|^{2+\varepsilon}$ at ∞ for some constants $C > 0$ and $\varepsilon > 0$, then equation (1.1) has infinitely many bounded solutions in $\mathbf{R}^n$ with positive lower bounds. Later on, Naito [12] improves the result: If $|K(x)| \leqslant \phi(|x|)$ for all $x \in \mathbf{R}^n$ and $\int_0^\infty t\phi(t)\,dt < \infty$, then equation (1.1) has infinite many bounded positive solutions which tend to a positive constant at ∞. On the other hand, when $K(x) \geqslant 0$, Ni [13] proves a nonexistence result: If $K(x) \geqslant C/|x|^{2-\varepsilon}$ at ∞ for some constants $C > 0$ and $\varepsilon > 0$, then (1.1) does not possess any positive solution in $\mathbf{R}^n$. Lin [9] proves that it is still true even $\varepsilon = 0$. In view of Naito's existence result, we expect that the following conjecture be true.

CONJECTURE. Let $K(x) \geqslant \tilde{K}(|x|) \geqslant 0$ for all $x \in \mathbf{R}^n$ and $\int_0^\infty s\tilde{K}(s)\,ds = \infty$. Then (1.1) does not possess any positive solution in $\mathbf{R}^n$.

We give three theorems which almost answer this conjecture completely. Following Ni [13], we define the averages of $u(x) > 0$ and $K(x) \geqslant 0$ by $\bar{u}(r)$ and $\overline{K}(r)$,

$$\bar{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x)\,dS, \tag{2.1}$$

$$\overline{K}(r) = \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)^{\mu/\sigma}} \right)^{-\sigma/\mu}, \tag{2.2}$$

where dS denotes the volume element in the surface integral, ω_n denotes the surface area of the unit sphere in $\mathbf{R}^n$ and $1/\mu + 1/\sigma = 1$.

For the sake of completeness, we give another proof of Lin's result of nonexistence [9] in the following.

THEOREM 2.1. *Let $K(x)$ be a locally Hölder continuous function. If $K(x) \geqslant 0$ and $\overline{K}(r) \geqslant C/r^2$ for r large for some constant $C > 0$, then equation* (1.1) *does not possess any positive solution in* $\mathbf{R}^n$.

PROOF. Let u be a positive solution of (1.1) in $\mathbf{R}^n$. Then from Ni [12, Lemma 3.21], we have

$$\begin{cases} \bar{u}''(r) + \dfrac{n-1}{r}\bar{u}'(r) \geqslant \bar{K}(r)\bar{u}^\sigma(r) & \text{in } (0,\infty), \\ \bar{u}(0) = \alpha > 0, \quad \bar{u}'(0) = 0. \end{cases} \tag{2.3}$$

Hence we have

$$\bar{u}(r) \geqslant \alpha + \frac{1}{n-2}\int_0^r s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds. \tag{2.4}$$

Now assume that $\bar{K}(r) \geqslant C/r^2$ for $r \geqslant R_0$. Let $r > R_0$. Then from (2.4), we have

$$\begin{aligned} \bar{u}(r) &\geqslant \alpha + \frac{1}{n-2}\int_0^{R_0} s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds \\ &\quad + \frac{1}{n-2}\int_{R_0}^r s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds \\ &\geqslant \alpha + \frac{1}{n-2}\int_{R_0}^r s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds \\ &\geqslant \alpha + \frac{\alpha^\sigma}{n-2}\cdot C\cdot\left[1 - \left(\frac{1}{2}\right)^{n-2}\right]\cdot\int_{R_0}^{r/2}\frac{1}{s}\,ds \\ &\geqslant C_1 \log r \end{aligned} \tag{2.5}$$

for some $C_1 > 0$ and $r \geqslant R_1 > 2R_0$. For $R > R_1$ and $R \leqslant s \leqslant r \leqslant 2R$, we have

$$1/2 \leqslant s/r \leqslant 1. \tag{2.6}$$

Hence

$$s\left[1 - \left(\frac{s}{r}\right)^{n-2}\right] = \frac{s}{r^{n-2}}[r^{n-2} - s^{n-2}] \geqslant (n-2)\left(\frac{1}{2}\right)^{n-2}(r-s). \tag{2.7}$$

From (2.4), (2.5) and (2.7), we obtain

$$\bar{u}(r) \geqslant C_1 \log R + \frac{C_2}{R^2}\int_R^r (r-s)\bar{u}^\sigma(s)\,ds \tag{2.8}$$

for $R > R_1$ and $R \leqslant r \leqslant 2R$, where $C_2 > 0$ is a constant. Let

$$g(r) = C_1 \log R + \frac{C_2}{R^2}\int_R^r (r-s)\bar{u}^\sigma(s)\,ds. \tag{2.9}$$

Then

$$\begin{aligned} g(R) &= C_1 \log R, \quad g'(R) = 0, \\ g'(r) &= \frac{C_2}{R^2}\int_R^r \bar{u}^\sigma(s)\,ds \geqslant 0, \end{aligned} \tag{2.10}$$

and

$$g''(r) = \frac{C_2}{R^2}\bar{u}^\sigma(r) \geqslant \frac{C_2}{R^2}(g(r))^\sigma. \tag{2.11}$$

From (2.10) and (2.11), we have

$$2g''(r)g'(r) \geqslant \frac{2C_2}{R^2}(g(r))^\sigma g'(r),$$

or

$$\frac{d}{dr}\left\{[g'(r)]^2\right\} \geqslant \frac{2C_2}{R^2}\frac{d}{dr}\left[\frac{1}{\sigma+1}g^{\sigma+1}(r)\right].$$

Hence

$$[g'(r)]^2 \geqslant \left(\frac{2C_2}{(\sigma+1)R^2}\right)\left[g^{\sigma+1}(r) - g^{\sigma+1}(R)\right]. \tag{2.12}$$

Let $\beta = C_1 \log R = g(R)$ and $\delta = C_2/R^2$. Then we have

$$[g'(r)]^2 \geqslant \frac{2\delta}{\sigma+1}\left[g^{\sigma+1}(r) - \beta^{\sigma+1}\right].$$

Thus

$$\int_\beta^{g(r)} \frac{dg}{\sqrt{g^{\sigma+1} - \beta^{\sigma+1}}} \geqslant \left(\frac{2\delta}{\sigma+1}\right)^{1/2}\int_R^r ds. \tag{2.13}$$

Let $g(r) = \beta z$, we have

$$\int_1^z \frac{dz'}{\sqrt{(z')^{\sigma+1} - 1}} \geqslant \left(\frac{2\delta}{\sigma+1}\right)^{1/2}\beta^{(\sigma-1)/2}(r-R). \tag{2.14}$$

Now if we choose R so large that

$$\begin{aligned}\left(\frac{2\delta}{\sigma+1}\right)^{1/2}\cdot\beta^{(\sigma-1)/2}\cdot R &= \left(\frac{2C_2}{(\sigma+1)R^2}\right)^{1/2}(C_1\log R)^{(\sigma-1)/2}\cdot R\\ &= \left(\frac{2C_2}{\sigma+1}\right)^{1/2}(C_1\log R)^{(\sigma-1)/2}\\ &> \int_1^\infty \frac{dz}{\sqrt{z^{\sigma+1}-1}}.\end{aligned} \tag{2.15}$$

Then there is a $R_c \leqslant 2R$, such that

$$\lim_{r\to R_c} g(r) = \infty. \tag{2.16}$$

But $u(R_c) \geqslant g(R_c) = \infty$. This is a contradiction. This completes the proof of this theorem.

Now we can state our main nonexistence results.

THEOREM 2.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function. If $\overline{K}(r)$ satisfies*

(1) *there exist $\alpha > 0$, $R_0 > 0$ and $C > 0$, such that*

$$\overline{K}(r) \geqslant C/r^\alpha \quad \textit{for } r \geqslant R_0,$$

(2) *there exist* $\varepsilon > 0$ *and* $P > 2$, *such that*

$$\int_R^{(P-1)R} r\bar{K}(r)\,dr \geqslant \varepsilon \quad \text{for } R \geqslant R_0,$$

then equation (1.1) *does not possess any positive solution in* $\mathbf{R}^n$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^n$. Then as in the proof of Theorem 2.1, we have

$$\bar{u}(r) \geqslant \alpha + \frac{1}{n-2}\int_0^r s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds. \tag{2.17}$$

From assumption (2), we have

$$\int_0^\infty s\bar{K}(s)\,ds = \infty. \tag{2.18}$$

Hence

$$\bar{u}(r) \geqslant \alpha + C\int_0^{r/2} \alpha^\sigma s\bar{K}(s)\,ds$$

and

$$\lim_{r\to\infty} \bar{u}(r) = \infty. \tag{2.19}$$

Thus we can choose R_0 so large that

$$\bar{u}(R_0) \geqslant 1. \tag{2.20}$$

Now let $R \geqslant R_0$. From assumption (2), we have

$$\begin{aligned} \bar{u}(PR) &\geqslant \bar{u}(R) + \frac{1}{n-2}\int_R^{PR} s\bar{K}(s)\left[1 - \left(\frac{s}{PR}\right)^{n-2}\right]\bar{u}^\sigma(R)\,ds \\ &\geqslant \bar{u}(R) + \frac{1}{n-2}\cdot\bar{u}^\sigma(R)\cdot\left[1 - \left(\frac{P-1}{P}\right)^{n-2}\right]\cdot\int_R^{(P-1)R} s\bar{K}(s)\,ds \\ &\geqslant \bar{u}(R) + C_1 u^\sigma(R), \end{aligned} \tag{2.21}$$

where $1 > C_1 > 0$ and C_1 is a constant.

From (2.20), (2.21) and the fact that $\sigma > 1$, we have

$$\bar{u}(P^m R) \geqslant (1 + C_1)^m \quad \text{for all } R \geqslant R_0 \text{ and } m \geqslant 1. \tag{2.22}$$

Choose $\alpha_1 > 0$ so small that

$$\log(1 + C_1) \geqslant \alpha_1[\log P + \log(PR_0)]. \tag{2.23}$$

Then

$$m\log(1 + C_1) \geqslant \alpha_1[m\log P + \log(PR_0)]. \tag{2.24}$$

Hence $(1 + C_1)^m \geqslant (P^m R)^{\alpha_1}$ for all $m \geqslant 1$ and $PR_0 \geqslant R \geqslant R_0$. This means that $\bar{u}(P^m R) \geqslant (P^m R)^{\alpha_1}$ for all $m \geqslant 1$ and $PR_0 \geqslant R \geqslant R_0$. Hence

$$\bar{u}(r) \geqslant r^{\alpha_1} \quad \text{for } r \geqslant R_0. \tag{2.25}$$

Now we return to (2.21). We have for $R \geqslant R_0$

$$\begin{aligned} \bar{u}(P^m R) &\geqslant C_1\bar{u}^\sigma(P^{m-1}R) \geqslant C_1^{(1+\sigma+\cdots+\sigma^{m-1})}\cdot\bar{u}^{\sigma^m}(R) \\ &= C_1^{(\sigma^m-1)/(\sigma-1)}\cdot\bar{u}^{\sigma^m}(R), \qquad m \geqslant 1. \end{aligned} \tag{2.26}$$

Hence

$$\log(\bar{u}(P^m R)) \geqslant \sigma^m\left[\log \bar{u}(R) + \frac{1 - 1/\sigma^m}{\sigma - 1}\log C_1\right] \tag{2.27}$$

$$\geqslant \sigma^m\left[\alpha_1 \log R - \frac{1}{\sigma - 1}|\log C_1|\right].$$

Choose $C_2 > 0$ and R_1 sufficiently large, such that

$$\alpha_1 \log R_1 \geqslant \frac{1}{\sigma - 1}|\log C_1| + C_2. \tag{2.28}$$

Then

$$\log(\bar{u}(P^m R)) \geqslant C_2 \sigma^m \tag{2.29}$$

for $R \geqslant R_1$ and $m \geqslant 1$.

Now we can choose α_2 sufficiently small, such that

$$\log \sigma \geqslant \alpha_2(\log P + \log PR_1).$$

Then

$$m \log \sigma \geqslant \alpha_2(m \log P + \log PR_1), \qquad m \geqslant 1.$$

Hence $\sigma^m \geqslant (P^m R)^{\alpha_2}$ for $m \geqslant 1$ and $PR_1 \geqslant R \geqslant R_1$. Hence from (2.29), we have

$$\bar{u}(P^m R) \geqslant \exp\left[C_2 (P^m R)^{\alpha_2}\right]$$

for $m \geqslant 1$ and $PR_1 \geqslant R \geqslant R_1$. That is,

$$\bar{u}(r) \geqslant \exp\left[C_2 r^{\alpha_2}\right] \tag{2.30}$$

for $r \geqslant R_1$. Hence from (2.17), for $r \geqslant R_1$, we have

$$\bar{u}(r) \geqslant \bar{u}(R_1) + \frac{1}{n-2}\int_{R_1}^{r} s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^{\sigma}(s)\,ds$$

$$= \bar{u}(R_1) + \frac{1}{n-2}\int_{R_1}^{r} s\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\left[\bar{K}(s)\cdot \bar{u}^{(\sigma-1)/2}(s)\right]\bar{u}^{(\sigma+1)/2}(s)\,ds.$$

Now from (2.30) and the assumption (1), we can choose $R_2 \geqslant R_1$ so large that

$$\bar{K}(s)\bar{u}^{(\sigma-1)/2}(s) \geqslant C_3/s^2$$

for $s \geqslant R_2$ for some constant $C_3 > 0$. Hence we have

(2.31)

$$\bar{u}(r) \geqslant \bar{u}(R_1) + \frac{1}{n-2}\int_{R_2}^{r} s\left[\bar{K}(s)\cdot \bar{u}^{(\sigma-1)/2}(s)\right]\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^{(\sigma+1)/2}(s)\,ds$$

$$\geqslant \bar{u}(R_2) + \frac{1}{n-2}\int_{R_2}^{r} s\cdot\frac{C_3}{s^2}\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^{(\sigma+1)/2}(s)\,ds.$$

But from the proof of Theorem 2.1, this is impossible. Hence we complete the proof of this theorem.

THEOREM 2.3. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function. If $\bar{K}(r)$ satisfies*

(1) *$\int_0^r s\bar{K}(s)\,ds$ is strictly increasing in* $[0, \infty)$ *and $\int_0^\infty s\bar{K}(s)\,ds = \infty$,*

(2) *$(s/r)^m \leqslant \int_0^s t\bar{K}(t)\,dt / \int_0^r t\bar{K}(t)\,dt$ for some finite $m > 0$ and for all $r \geqslant s \geqslant R_0 > 0$,*

then equation (1.1) *does not possess any positive solution in* $\mathbf{R}^n$.

In particular, if $\bar{K}(r)$ satisfies (1) *and* $0 \leqslant \bar{K}(r) \leqslant C/r^2$ *for* $r \geqslant R_1$ *for some constants* $C > 0$ *and* $R_1 > 0$, *then* $\bar{K}(r)$ *also satisfies* (2) *and hence* (1.1) *does not possess any positive solution in* $\mathbf{R}^n$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^n$. Then as in the proof of Theorem 2.2, we have (2.17). Let

$$f(r) = \int_0^r s\bar{K}(s)\,ds = \eta.$$

Then f: $[0, \infty) \to [0, \infty)$ is one-one and onto. Hence f^{-1} exists and let it be denoted by g. Let

$$t = f(s), \quad \eta = f(r), \quad \bar{u}(g(\eta)) = v(\eta).$$

Then from (2.17), we have

$$v(\eta) \geqslant \alpha + \frac{1}{n-2}\int_0^\eta \left[1 - \left(\frac{g(t)}{g(\eta)}\right)^{(n-2)}\right] v^\sigma(t)\,dt. \tag{2.32}$$

From the assumption (2), we have

$$g(t)/g(\eta) \leqslant (t/\eta)^{1/m} \quad \text{for all } \eta \geqslant t \geqslant f(R_0). \tag{2.33}$$

Hence from (2.32) and (2.33), we have

$$v(\eta) \geqslant \bar{u}(R_0) + \frac{1}{n-2}\int_{f(R_0)}^\eta \left[1 - \left(\frac{t}{\eta}\right)^{(n-2)/m}\right] v^\sigma(t)\,dt. \tag{2.34}$$

But from Theorem 2.1, this is impossible. Hence (1.1) does not possess any positive solution.

If in addition to condition (1), $\bar{K}(r)$ also satisfies $0 \leqslant \bar{K}(r) \leqslant C/r^2$ for $r \geqslant R_1$. Then we have

$$\frac{d}{dr}\left(\frac{\int_0^r t\bar{K}(t)\,dt}{r}\right) = \frac{r^2\bar{K}(r) - \int_0^r t\bar{K}(t)\,dt}{r^2} \leqslant \frac{C - \int_0^r t\bar{K}(t)\,dt}{r^2.}$$

for $r \geqslant R_1$. Thus we can choose $R_2 \geqslant R_1$ so large that

$$C - \int_0^r t\bar{K}(t)\,dt \leqslant 0 \quad \text{for } r \geqslant R_2.$$

Hence $\int_0^r t\bar{K}(t)\,dt/r$ is monotonically decreasing for $r \geqslant R_2$. Thus $\bar{K}(r)$ satisfies condition (2) for $r \geqslant s \geqslant R_2$.

This completes the proof of this theorem.

THEOREM 2.4. *Let* $K(x) \geqslant 0$ *be a locally Hölder continuous function in* $\mathbf{R}^n$ *and* $\tilde{K}(t)$ *be a locally Hölder continuous function in* $[0, \infty)$.

Let the average $\bar{K}(r)$ *of* $K(x)$ *in the sense of* (2.2) *satisfy*:

$$\bar{K}(r) \geqslant \tilde{K}(r - \beta_i) \quad \text{if } \alpha_i + \beta_i \leqslant r \leqslant \alpha_{i+1} + \beta_i,$$

$$\bar{K}(r) \geqslant 0 \quad \text{if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1}$$

for $i = 0, 1, 2, \ldots$, *where* $\{\alpha_i\}_{i=0}^{\infty}$ *is a strictly increasing sequence satisfying* $\alpha_0 = 0$ *and* $\lim_{n\to\infty} \alpha_n = \infty$ *and* $\{\beta_i\}_{i=0}^{\infty}$ *is a nondecreasing sequence satisfying* $\beta_0 = 0$ *and* $\beta_i/\alpha_i \leqslant M$ *for some constant* $M > 0$ *and* $i = 1, 2, \ldots$. *If*

$$\begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) = \tilde{K}(r) u^{\sigma}(r) & \text{in } (0, \infty), \\ u(0) = \alpha > 0, \quad u'(0) = 0 \end{cases} \tag{2.35}$$

does not possess any solution in $[0, \infty)$ *for all* $\alpha > 0$, *then* (1.1) *does not possess any positive solution in* $\mathbf{R}^n$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^n$. Then as in the proof of Theorem 2.2, we have

$$\bar{u}(r) \geqslant \alpha + \frac{1}{n-2} \int_0^r s\bar{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds. \tag{2.36}$$

Now we define the function v by

$$v(r) = \bar{u}(r + \beta_i) \quad \text{if } \alpha_i \leqslant r < \alpha_{i+1} \tag{2.37}$$

for $i = 0, 1, 2, \ldots$. We shall prove that

$$v(r) \geqslant \alpha + \frac{A}{n-2} \int_0^r s\tilde{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right] v^{\sigma}(s)\, ds, \tag{2.38}$$

where A is a positive constant depending only on the constant M. To prove (2.38), let $\alpha_i \leqslant r \leqslant \alpha_{i+1}$. Then from (2.36), we have

$$\begin{aligned} \bar{u}(r + \beta_i) &\geqslant \alpha + \frac{1}{n-2} \int_0^{r+\beta_i} s\bar{K}(s)\left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds \\ &\geqslant \alpha + \frac{1}{n-2} \int_0^{\alpha_1} s\bar{K}(s)\left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds \\ &\quad + \frac{1}{n-2} \int_{\alpha_1+\beta_1}^{\alpha_2+\beta_1} s\bar{K}(s)\left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds \\ &\quad + \cdots \\ &\quad + \frac{1}{n-2} \int_{\alpha_i+\beta_i}^{r+\beta_i} s\bar{K}(s)\left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds \\ &= \alpha + \frac{1}{n-2} \int_0^{\alpha_1} s\bar{K}(s)\left[1 - \left(\frac{s}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s)\, ds \\ &\quad + \frac{1}{n-2} \int_{\alpha_1}^{\alpha_2} (s+\beta_1)\bar{K}(s+\beta_1)\left[1 - \left(\frac{s+\beta_1}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s+\beta_1)\, ds \\ &\quad + \cdots \\ &\quad + \frac{1}{n-2} \int_{\alpha_i}^{r} (s+\beta_i)\bar{K}(s+\beta_i)\left[1 - \left(\frac{s+\beta_i}{r+\beta_i}\right)^{n-2}\right] \bar{u}^{\sigma}(s+\beta_i)\, ds. \end{aligned}$$

But for $1 \leqslant j \leqslant i$,

$$1 - \left(\frac{s+\beta_j}{r+\beta_i}\right)^{n-2} \geqslant 1 - \left(\frac{s+\beta_i}{r+\beta_i}\right)^{n-2} = \frac{(1+\beta_i/r)^{n-2} - (s/r+\beta_i/r)^{n-2}}{(1+\beta_i/r)^{n-2}}$$

$$\geqslant \frac{1-(s/r)^{n-2}}{(1+\beta_i/\alpha_i)^{n-2}} \geqslant A\left[1-(s/r)^{n-2}\right].$$

Hence we have

$$\begin{aligned}\bar{u}(r+\beta_i) \geqslant \alpha &+ \frac{A}{n-2}\int_0^{\alpha_1} s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s)\,ds \\ &+ \frac{A}{n-2}\int_{\alpha_1}^{\alpha_2} s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s+\beta_1)\,ds \\ &+ \cdots \\ &+ \frac{A}{n-2}\int_{\alpha_i}^{r} s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]\bar{u}^\sigma(s+\beta_i)\,ds.\end{aligned}$$

Hence (2.38) is true for all $r \in [0,\infty)$. Let $\tilde{v} = A^{1/(\sigma-1)}v$ and $\tilde{\alpha} = A^{1/(\sigma-1)}\alpha$. Then (2.38) becomes

$$\tilde{v}(r) \geqslant \tilde{\alpha} + \frac{1}{n-2}\int_0^r sK(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]\tilde{v}^\sigma(s)\,ds.$$

Now let X denote the locally convex space of all continuous functions on $[0,\infty)$ with the usual topology and consider the set

$$Y = \{y \in X: \tilde{\alpha} \leqslant y(r) \leqslant \tilde{v}(r) \text{ for } r \geqslant 0\},$$

where $\tilde{v}$ is defined above. Clearly, Y is a closed convex subset of X. Define the mapping T by

$$Ty(r) = \tilde{\alpha} + \frac{1}{n-2}\int_0^r s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n+2}\right]y^\sigma(s)\,ds. \tag{2.39}$$

If $y \in Y$, then $\tilde{\alpha} \leqslant y(r) \leqslant \tilde{v}(r)$. Hence we have

$$Ty(r) = \tilde{\alpha} + \frac{1}{n-2}\int_0^r s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]y^\sigma(s)\,ds \geqslant \tilde{\alpha}$$

and

$$Ty(r) \leqslant \tilde{\alpha} + \frac{1}{n-2}\int_0^r s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]\tilde{v}^\sigma(s)\,ds \leqslant \tilde{v}(r).$$

Thus T maps Y into itself. Let $\{y_m\}_{m=1}^\infty \subset Y$ be a sequence which converges to y in X. Then $\{y_m\}$ converges uniformly to y on any compact interval of $[0,\infty)$. Since

$$|Ty_m(r) - Ty(r)| \leqslant \frac{1}{n-2}\int_0^r s\tilde{K}(s)\left[1-\left(\frac{s}{r}\right)^{n-2}\right]|y_m^\sigma(s) - y^\sigma(s)|\,ds, \tag{2.40}$$

we have $\{Ty_m\}$ converges uniformly to Ty on any compact interval of $[0,\infty)$. Hence T is a continuous mapping from Y into Y. On the other hand, we have

$$(Ty)'(r) = \int_0^r \left(\frac{s}{r}\right)^{n-1}\tilde{K}(s)y^\sigma(s)\,ds. \tag{2.41}$$

Hence for any fixed $R > 0$, TY is a uniformly bounded and equicontinuous family of functions defined on $[0, R]$. Hence TY is relatively compact. Thus we can use the Schauder-Tychonoff fixed point theorem (see Edwards [**2**, p. 161]) to conclude that T has a fixed point $y \in Y$. This fixed point y satisfies the integral equation

$$y(r) = \tilde{\alpha} + \frac{1}{n-2}\int_0^r s\tilde{K}(s)\left[1 - \left(\frac{s}{r}\right)^{n-2}\right]y^\sigma(s)\,ds.$$

Hence (2.35) has a solution for this $\tilde{\alpha}$. This is a contradiction. The theorem is proved. Q.E.D.

3. The case $n = 2$. In this case, we consider only the situation $K(x) \geqslant 0$ in (1.1). Kawano, Kusano and Naito [**3**] obtain the following existence result: Let $K(x) \geqslant 0$ be a locally Hölder continuous function which is positive in some neighborhood of the origin. If

$$K(x) \leqslant \tilde{K}(|x|) \quad \text{for all } x \in \mathbf{R}^2$$

and

$$\int_1^\infty s(\log s)^\sigma \tilde{K}(s)\,ds < \infty.$$

Then equation (1.1) has infinitely many positive solutions in $\mathbf{R}^2$ with logarithmic growth at infinity.

To our knowledge, there seems no known nonexistence result. Our nonexistence results are

THEOREM 3.1. *Let* $K(x) \geqslant 0$ *be a locally Hölder continuous function in* $\mathbf{R}^2$. *Let the average* $\overline{K}(r)$ *of* $K(x)$ *in the sense of* (2.2) *satisfy*

$$\overline{K}(r) \geqslant C/r^2(\log r)^{\sigma+1} \quad \textit{for } r \geqslant R_0. \tag{3.1}$$

Then equation (1.1) *does not possess any positive solution in* $\mathbf{R}^2$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^2$. Then we have

$$\begin{cases} \bar{u}''(r) + \bar{u}'(r)/r \geqslant \overline{K}(r)\bar{u}^\sigma(r), \\ \bar{u}(0) = \alpha > 0, \quad \bar{u}'(0) = 0, \end{cases} \tag{3.2}$$

where $\bar{u}$ and $\overline{K}$ are defined in (2.1) and (2.2). From (3.2), $\bar{u}(r)$ satisfies the integral equation

$$\bar{u}(r) \geqslant \alpha + \int_0^r s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds. \tag{3.3}$$

Without loss of generality, we assume that $K(0) > 0$ and hence $\overline{K}(0) > 0$. Thus we have from (3.3)

$$\begin{aligned} \bar{u}(r) &\geqslant \alpha + \int_0^1 s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds + \int_1^r s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds \\ &\geqslant \alpha + \int_0^1 s\log r\overline{K}(s)\bar{u}^\sigma(s)\,ds \\ &\geqslant \alpha + \alpha^\sigma \cdot \log r \cdot \int_0^1 s\overline{K}(s)\,ds \\ &\geqslant \alpha + C_1 \log r \end{aligned} \tag{3.4}$$

for $r \geqslant 1$ and a constant $C_1 > 0$.

Now consider $r \geqslant e$. We have

$$\bar{u}(r) \geqslant \alpha + \int_0^1 s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds + \int_e^r s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds \geqslant C_1\log r + \int_e^r s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds. \tag{3.5}$$

Let $v(r) = \bar{u}(r)/\log r$ for $r \geqslant e$. Then from (3.5), we have

$$v(r) \geqslant C_1 + \int_e^r s\left(1 - \frac{\log s}{\log r}\right)\overline{K}(s)(\log s)^\sigma v^\sigma(s)\,ds. \tag{3.6}$$

Let $t = \log s$, $\eta = \log r$ and $v(e^\eta) = v(r) = \tilde{v}(\eta)$. Then (3.6) becomes

$$\tilde{v}(\eta) \geqslant C_1 + \int_1^\eta t\left(1 - \frac{t}{\eta}\right)e^{2t}\overline{K}(e^t)t^{(\sigma-1)}\tilde{v}^\sigma(t)\,dt. \tag{3.7}$$

Let $\tilde{K}(t) = e^{2t}\overline{K}(e^t)t^{(\sigma-1)}$. Then from (3.1), we have

$$\tilde{K}(t) \geqslant C/t^2 \quad \text{for } t \geqslant \exp(R_0)$$

and

$$\tilde{v}(\eta) \geqslant C_1 + \int_1^\eta t\left(1 - \frac{t}{\eta}\right)\tilde{K}(t)\tilde{v}^\sigma(t)\,dt. \tag{3.8}$$

Using a similar argument as in the proof of Theorem 2.1, we obtain a contradiction. This completes the proof of this theorem. Q.E.D.

THEOREM 3.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of* (2.2) *satisfy*

(3.9) *There exist $\varepsilon > 0$, $P > 2$ and $R_0 > 0$, such that*

$$\int_{e^R}^{e^{(P-1)R}} s\overline{K}(s)(\log s)^\sigma ds \geqslant \varepsilon \quad \textit{for all } R \geqslant R_0.$$

(3.10) *There exist $\alpha > 0$, $R_1 > 0$ and $C > 0$, such that*

$$\overline{K}(s) \geqslant C/s^2(\log s)^{(\sigma+\alpha)} \quad \textit{for all } s \geqslant R_1.$$

Then equation (1.1) *does not possess any positive solution in* $\mathbf{R}^2$.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^2$. As in the proof of Theorem 3.1, we have (3.3)–(3.7). Hence

$$\tilde{v}(\eta) \geqslant C_1 + \int_1^\eta t\left(1 - \frac{t}{\eta}\right)\tilde{K}(t)\tilde{v}^\sigma(t)\,dt. \tag{3.11}$$

But from (3.9) and (3.10), $\tilde{K}(t)$ satisfies

$$\int_R^{(P-1)R} t\tilde{K}(t)\,dt \geqslant \varepsilon \quad \text{for all } R \geqslant R_0, \tag{3.12}$$

$$\tilde{K}(s) \geqslant C/t^{(1+\alpha)} \quad \text{for all } t \geqslant \log R_1. \tag{3.13}$$

Using a similar argument as in the proof of Theorem 2.2, we obtain a contradiction. This completes the proof. Q.E.D.

THEOREM 3.3. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of* (2.2) *satisfy*

$$\int_0^r s\overline{K}(s)(\log s)^\sigma\,ds \text{ is strictly increasing on } [0,\infty) \text{ and} \tag{3.14}$$

$$\int_0^\infty s\overline{K}(s)(\log s)^\sigma\,ds = \infty,$$

$$\left(\frac{\log s}{\log r}\right)^m \leqslant \int_0^s t\overline{K}(t)(\log t)^\sigma\,dt \Big/ \int_0^r t\overline{K}(t)(\log t)^\sigma\,dt \tag{3.15}$$

for some $m > 0$ and for all $r \geqslant s \geqslant R_0 > 0$. Then equation (1.1) *does not possess any positive solution in $\mathbf{R}^2$. In particular, if $\overline{K}(r)$ satisfies* (3.14) *and $0 \leqslant \overline{K}(r) \leqslant C/r^2(\log r)^{\sigma+1}$ for $r \geqslant R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\overline{K}(r)$ also satisfies* (3.15) *and hence* (1.1) *does not possess any positive solution in $\mathbf{R}^2$.*

PROOF. Assume that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^2$. As in the proof of Theorem 3.1, we have (3.3)–(3.7). Hence we obtain (3.8) or (3.11). But now $\tilde{K}(t)$ satisfies

$$\int_1^\infty t\tilde{K}(t)\,dt \text{ is strictly increasing in } [1,\infty) \text{ and} \tag{3.15}$$

$$\int_1^\infty t\tilde{K}(t)\,dt = \infty,$$

$$\left(\frac{s}{\eta}\right)^m \leqslant \int_1^s t\tilde{K}(t)\,dt \Big/ \int_1^\eta t\tilde{K}(t)\,dt \tag{3.16}$$

for some $m > 0$ and for all $\eta \geqslant s \geqslant \log R_0$.

Using a similar argument as in the proof of Theorem 2.3, we obtain a contradiction. This completes the proof. Q.E.D.

THEOREM 3.4. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$ and $\tilde{K}(t)$ be a locally Hölder continuous function in $[0,\infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of* (2.2) *satisfy*

$$\overline{K}(r) \geqslant 0 \quad \text{if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1},$$
$$\overline{K}(r) \geqslant \tilde{K}(r - \beta_i) \quad \text{if } \alpha_i + \beta_i \leqslant r \leqslant \alpha_{i+1} + \beta_i$$

for $i = 0, 1, 2, \ldots$, where $\{\alpha_i\}_{i=0}^\infty$ is a strictly increasing sequence satisfying $\alpha_0 = 0$ and $\lim_{n\to\infty} \alpha_n = \infty$ and $\{\beta_i\}_{i=0}^\infty$ is a nondecreasing sequence satisfying $\beta_0 = 0$ and $\beta_i/\alpha_i \leqslant M$ for some $M > 0$ for all $i \geqslant 1$. If

$$\begin{cases} u''(r) + u'(r)/r = \tilde{K}(r)u^\sigma(r) & \text{in } (0,\infty), \\ u(0) = \alpha > 0, \quad u'(0) = 0 \end{cases} \tag{3.17}$$

does not possess any solution in $[0,\infty)$ for all $\alpha > 0$, then (1.1) *does not possess any positive solution in $\mathbf{R}^2$.*

PROOF. The proof is very similar to that of Theorem 2.4. Hence we only sketch the proof. Assume that (1.1) has a positive solution in $\mathbf{R}^2$. Then we have

$$\bar{u}(r) \geqslant \alpha + \int_0^r s\log\left(\frac{r}{s}\right)\overline{K}(s)\bar{u}^\sigma(s)\,ds. \tag{3.18}$$

Let

$$v(r) = \bar{u}(r+\beta_i) \quad \text{if } \alpha_i \leqslant r < \alpha_{i+1}$$

for $i = 0, 1, 2, \ldots$. Then

$$v(r) \geqslant \alpha + A \cdot \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}(s) v^\sigma(s)\, ds. \tag{3.19}$$

Let X denote the locally convex space of all continuous function on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ y \in X \colon \tilde{\alpha} \leqslant y(r) \leqslant \tilde{v}(r) \text{ for } r \geqslant 0 \}.$$

Define the mapping T by

$$(Ty)(r) = \tilde{\alpha} + \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}(s) y^\sigma(s)\, ds. \tag{3.20}$$

We can prove that $TY \subset Y$ and T is continuous. Furthermore TY is relatively compact. Hence T has a fixed point in Y. Thus (3.17) has a solution for this given $\tilde{\alpha} > 0$. This is a contradiction. The proof is complete. Q.E.D.

4. The case $n = 1$. In this case, we also consider only the situation $K(x) \geqslant 0$ in (1.1). We give a main existence result which have an extension to the higher-dimensional case. We also give some nonexistence results which may have applications.

THEOREM 4.1. *Let $K(x) \geqslant 0$ be a Hölder continuous (actually only continuous is sufficient) function in* $\mathbf{R}$. *If $K(0) > 0$*

$$\int_{-\infty}^{\infty} |x|^\sigma K(x)\, dx < \infty, \tag{4.1}$$

then (1.1) *has infinitely many positive solutions in* $\mathbf{R}$ *with linear growth at* $|x| = \infty$.

PROOF. We shall seek solutions u such that $u(0) = \alpha > 0$ and $u'(0) = 0$. Consider now $x \geqslant 0$. Then equation (1.1) with $u(0) = \alpha > 0$ and $u'(0) = 0$ is equivalent to the integral equation

$$u(x) = \alpha + \int_0^x (x-t) K(t) u^\sigma(t)\, dt, \qquad x \geqslant 0. \tag{4.2}$$

Now choose α so small that

$$2^\sigma \alpha^{(\sigma-1)} \int_0^1 K(t)\, dt \leqslant \frac{1}{2}, \tag{4.3}$$

$$2^\sigma \alpha^{(\sigma-1)} \int_1^\infty K(t) t^\sigma\, dt \leqslant \frac{1}{2}. \tag{4.4}$$

Let

$$A(x) = \begin{cases} 2\alpha & \text{if } 0 \leqslant x \leqslant 1, \\ 2\alpha x & \text{if } 1 \leqslant x. \end{cases}$$

As in the proofs of Theorems 2.4 and 3.4, we let X denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ y \in X \colon \alpha \leqslant y(x) \leqslant A(x) \text{ for } x \geqslant 0 \}.$$

Clearly, Y is a closed convex subset of X. Let the mapping T be defined by

$$(4.5)\qquad (Ty)(x) = \alpha + \int_0^x (x-t)K(t)y^\sigma(t)\,dt, \qquad x \geqslant 0.$$

If $y \in Y$, then $\alpha \leqslant y(x) \leqslant A(x)$. Hence we have

$$(4.6)\qquad \begin{aligned}(Ty)(x) &= \alpha + \int_0^x (x-t)K(t)y^\sigma(t)\,dt \\ &\geqslant \alpha + \int_0^x (x-t)K(t)\alpha^\sigma dt \geqslant \alpha.\end{aligned}$$

On the other hand, for $0 \leqslant x \leqslant 1$, we have

$$(4.7)\qquad \begin{aligned}(Ty)(x) &= \alpha + \int_0^x (x-t)K(t)y^\sigma(t)\,dt \\ &\leqslant \alpha + \int_0^1 K(t)(2\alpha)^\sigma\,dt \\ &= \alpha\left[1 + 2^\sigma\alpha^{(\sigma-1)}\int_0^1 K(t)\,dt\right] \\ &\leqslant \alpha\left[1 + \tfrac{1}{2}\right] \leqslant 2\alpha = A(x).\end{aligned}$$

For $1 \leqslant x$, we have

$$(4.8)\qquad \begin{aligned}(Ty)(x) &= \alpha + \int_0^1 (x-t)K(t)y^\sigma(t)\,dt + \int_1^x (x-t)K(t)y^\sigma(t)\,dt \\ &\leqslant \alpha + x\int_0^1 K(t)(2\alpha)^\sigma\,dt + x\int_1^\infty K(t)(2\alpha t)^\sigma\,dt \\ &\leqslant \alpha x + \alpha x\left[2^\sigma\alpha^{(\sigma-1)}\int_0^1 K(t)\,dt\right] + \alpha x\left[2^\sigma\alpha^{(\sigma-1)}\int_1^\infty K(t)t^\sigma\,dt\right] \\ &\leqslant \alpha x\left[1 + \tfrac{1}{2} + \tfrac{1}{2}\right] \leqslant 2\alpha x = A(x).\end{aligned}$$

Thus T maps Y into itself. Now let $\{y_m\}_{m=1}^\infty \subset Y$ be a sequence which converges to y in X. Then $\{y_m\}$ converges uniformly to y on any compact interval of $[0, \infty)$. But

$$(4.9)\qquad |Ty_m(x) - Ty(x)| \leqslant \int_0^x (x-t)K(t)|y_m^\sigma(t) - y^\sigma(t)|\,dt,$$

we conclude that $\{Ty_m\}$ converges uniformly to Ty on any compact interval of $[0, \infty)$. Hence T is a continuous mapping from Y into Y. As in the proof of Theorem 2.4, the precompactness of T can be verified by

$$(4.10)\qquad \begin{aligned}|(Ty)'(x)| &\leqslant \int_0^x K(t)y^\sigma(t)\,dt \\ &\leqslant \int_0^\infty K(t)(2\alpha)^\sigma t^\sigma\,dt < \infty.\end{aligned}$$

Thus T has a fixed point $y \in Y$. This fixed point y is a solution of equation (1.1) for $x \geqslant 0$ with $y(0) = \alpha$ and $y'(0) = 0$.

Similarly, we can find a solution of equation (1.1) for $x \leqslant 0$ with $y(0) = \alpha$ and $y'(0) = 0$ if α is sufficiently small. Now let $y(x)$ be the solution of (1.1) in $\mathbf{R}$ with

$y(0) = \alpha,\ y'(0) = 0$. Then

$$2\alpha x \geqslant y(x) = \alpha + \int_0^x (x-t)K(y)y^\sigma(t)\,dt \geqslant \alpha + \int_0^1 (x-1)K(t)\alpha^\sigma dt \geqslant \alpha + k_1(x-1) \geqslant k_2 x \tag{4.11}$$

for x large. Hence y grows linearly at $|x| = \infty$. Now we can choose a smaller $y(0)$, such as $y(0) = \alpha/2$ to obtain another solution. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [**13, 14**] and Kawano, Kusano and Naito [**3**].

THEOREM 4.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$. Let $\phi_*(x_1)$ and $\phi^*(x_1)$ be two locally Hölder continuous function in $\mathbf{R}$. If*

$$0 \leqslant \phi_*(x_1) \leqslant K(x) \leqslant \phi^*(x_1) \quad \textit{for all } x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}, \tag{4.12}$$

$$\phi_*(0) > 0 \quad \textit{and} \quad \int_{-\infty}^{\infty} |x_1|^\sigma \phi^*(x_1)\,dx_1 < \infty, \tag{4.13}$$

then equation (1.1) *has infinitely many positive solutions in $\mathbf{R}^n$ which are unbounded.*

PROOF. Consider the equations

$$d^2\tilde{v}/dx_1^2 = \phi^*(x_1)\tilde{v}^\sigma, \tag{4.14}$$

$$d^2\tilde{w}/dx_1^2 = \phi_*(x_1)\tilde{w}^\sigma. \tag{4.15}$$

From the proof of Theorem 4.1 we see that (4.14) and (4.15) have unbounded solutions (linear growth at ∞) $\tilde{v}$ and $\tilde{w}$. We can choose $\tilde{v}$ and $\tilde{w}$ such that $\tilde{v}(x_1) \leqslant \tilde{w}(x_1)$ for all $x_1 \in \mathbf{R}$. Now let

$$v(x_1, x') = \tilde{v}(x_1) \quad \text{and} \quad w(x_1, x') = \tilde{w}(x_1). \tag{4.16}$$

Then from (4.12), we have

$$\Delta v - K(x)v^\sigma = \frac{d^2\tilde{v}(x_1)}{dx_1^2} - K(x)\tilde{v}^\sigma(x_1) = [\phi^*(x_1) - K(x)]\,\tilde{v}^\sigma(x_1) \geqslant 0,$$

$$\Delta w - K(x)w^\sigma = \frac{d^2\tilde{w}(x_1)}{dx_1^2} - K(x)\tilde{w}^\sigma(x_1) = [\phi_*(x_1) - K(x)]\,\tilde{w}^\sigma(x_1) \leqslant 0$$

in $\mathbf{R}^n$. Hence $v(x_1, x')$ and $w(x_1, x')$ are, respectively, a subsolution and a supersolution of (1.1) in $\mathbf{R}^n$. Since $v(x_1, x') \leqslant w(x_1, x')$ in $\mathbf{R}^n$, from Theorem 2.10 of Ni [**13**], it follows that (1.1) has a positive solution $u(x)$ in $\mathbf{R}^n$ such that $\tilde{v}(x_1) \leqslant u(x_1, x') \leqslant \tilde{w}(x_1)$. It is easy to see that $k_1|x_1| \leqslant u(x_1, x') \leqslant k_2|x_1|$ for $|x_1|$ large for some positive constants k_1 and k_2. This completes the proof of the theorem. Q.E.D.

Now let u be a positive function in $\mathbf{R}$ and $K(x) \geqslant 0$ in $\mathbf{R}$. Define for $r \geqslant 0$

$$\bar{u}(r) = (u(r) + u(-r))/2, \tag{4.17}$$

$$\bar{K}(r) = \left[\tfrac{1}{2}\left(K(r)^{-\sigma'/\sigma} + K(-r)^{-\sigma'/\sigma}\right)\right]^{-\sigma/\sigma'} \tag{4.18}$$

where $1/\sigma + 1/\sigma' = 1$. It is easy to see that

$$\bar{u}(0) = u(0) \quad \text{and} \quad \bar{u}'(0) = 0 \tag{4.19}$$

if u is also continuously differentiable.

THEOREM 4.3. *Let $K(x) \geqslant 0$ be a continuous function in* $\mathbf{R}$. *If the average $\bar{K}(r)$ of $K(x)$ in the sense* (4.18) *satisfies*

$$\bar{K}(r) \geqslant C/r^{(\sigma+1)} \tag{4.20}$$

for $r \geqslant R_0$ for some constant $C > 0$, then equation (1.1) *does not possess any positive solution in* $\mathbf{R}$.

PROOF. Assume that $u(x)$ is a positive solution of (1.1) in $\mathbf{R}$. Then we have

$$\bar{u}''(r) = \frac{u''(r) + u''(-r)}{2} = \frac{1}{2}\left[K(r)u^\sigma(r) + K(-r)u^\sigma(-r)\right]. \tag{4.21}$$

But

$$\begin{aligned} \bar{u}(r) &= \tfrac{1}{2}[u(r) + u(-r)] \\ &\leqslant \left[\tfrac{1}{2}(K(r)u^\sigma(r) + K(-r)u^\sigma(-r))\right]^{1/\sigma} \\ &\quad \cdot \left[\tfrac{1}{2}(K^{-\sigma'/\sigma}(r) + K^{-\sigma'/\sigma}(-r))\right]^{1/\sigma'}. \end{aligned} \tag{4.22}$$

Hence

$$\tfrac{1}{2}(K(r)u^\sigma(r) + K(-r)u^\sigma(-r)) \geqslant \bar{K}(r)\bar{u}^\sigma(r). \tag{4.23}$$

Thus we have

$$\begin{cases} \bar{u}''(r) \geqslant \bar{K}(r)\bar{u}^\sigma(r) & \text{for } r > 0, \\ \bar{u}(0) = \alpha > 0, \quad \bar{u}'(0) = 0. \end{cases} \tag{4.24}$$

Hence $\bar{u}$ satisfies

$$\bar{u}(r) \geqslant \alpha + \int_0^r (r - t)\bar{K}(t)\bar{u}^\sigma(t)\,dt. \tag{4.25}$$

Without loss of generality, we may assume that $K(0) > 0$ and hence $\bar{K}(0) > 0$. Thus for $r \geqslant 2$, we have

$$\begin{aligned} \bar{u}(r) &\geqslant \alpha + \int_0^1 (r - t)\bar{K}(t)\bar{u}^\sigma(t)\,dt + \int_1^r (r - t)\bar{K}(t)\bar{u}^\sigma(t)\,dt \\ &\geqslant \alpha + \left(\alpha^\sigma \cdot \int_0^1 \left(1 - \frac{t}{r}\right)\bar{K}(t)\,dt\right) \cdot r + \int_1^r (r - t)\bar{K}(t)\bar{u}^\sigma(t)\,dt \\ &\geqslant C_1 \cdot r + \int_1^r (r - t)\bar{K}(t)\bar{u}^\sigma(t)\,dt, \end{aligned} \tag{4.26}$$

where

$$C_1 = \alpha^\sigma \cdot \int_0^1 \left(1 - \frac{1}{2}\right)\overline{K}(t)\,dt = \alpha^\sigma \cdot \frac{1}{2} \cdot \int_0^1 \overline{K}(t)\,dt > 0.$$

Now let $\bar{u}(r) = v(r) \cdot r$ for $r \geqslant 2$. We obtain

$$v(r) \geqslant C_1 + \int_1^r t\left(1 - \frac{t}{r}\right)\overline{K}(t)t^{(\sigma-1)}v^\sigma(t)\,dt. \tag{4.27}$$

Letting $\tilde{K}(t) = \overline{K}(t)t^{(\sigma-1)}$. Then from (4.20), we have

$$\tilde{K}(t) \geqslant C/t^2 \quad \text{for } t \geqslant R_0 \tag{4.28}$$

and

$$v(r) \geqslant C_1 + \int_1^r t\tilde{K}(t)\left(1 - \frac{t}{r}\right)v^\sigma(t)\,dt. \tag{4.29}$$

From the proof of Theorem 2.1, we see that it is impossible to have a function v defined in $[2, \infty)$ satisfying (4.29). This completes the proof. Q.E.D.

THEOREM 4.4. *Let $K(x) \geqslant 0$ be a continuous function in $\mathbf{R}$. If the average $\overline{K}(r)$ of $K(r)$ in the sense* (4.18) *satisfies*

(4.30) *there exist $\alpha > 0$, $R_0 > 0$ and $C > 0$ such that*

$$\overline{K}(r) \geqslant C/r^{(\sigma+\alpha)} \quad \textit{for } r \geqslant R_0,$$

(4.31) *there exist $\varepsilon > 0$ and $P > 2$ such that*

$$\int_R^{(P-1)R} r^\sigma \overline{K}(r)\,dr \geqslant \varepsilon \quad \textit{for } \mathbf{R} \geqslant R_0.$$

Then equation (1.1) *does not possess any positive solution in* $\mathbf{R}$.

PROOF. Assume on the contrary that (1.1) has a positive solution $u(x)$ in $\mathbf{R}$. Then as in the proof of Theorem 4.3, we have (4.24)–(4.27). But now $\tilde{K}(r) = r^{(\sigma-1)}\overline{K}(r)$ satisfies

$$\tilde{K}(r) \geqslant C/r^{(1+\alpha)} \quad \text{for } r \geqslant R_0, \tag{4.32}$$

$$\int_R^{(P-1)R} r\tilde{K}(r)\,dr \geqslant \varepsilon \quad \text{for } R \geqslant R_0. \tag{4.33}$$

But from the proof of Theorem 2.2, there is no positive function v satisfying (4.27). This contradiction proves the theorem. Q.E.D.

THEOREM 4.5. *Let $K(x) \geqslant 0$ be a continuous function in $\mathbf{R}$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense* (4.18) *satisfy*

(4.34) $\int_0^r s^\sigma \overline{K}(s)\,ds$ *is strictly increasing in* $[0, \infty)$ *and*

$$\int_0^\infty s^\sigma \overline{K}(s)\,ds = \infty,$$

(4.35) $\left(\frac{s}{r}\right)^m \leqslant \int_0^s t^\sigma \overline{K}(t)\,dt \Big/ \int_0^r t^\sigma \overline{K}(t)\,dt$ *for some $m > 0$ and for all $r \geqslant s \geqslant R_0 > 0$.*

Then equation (1.1) *does not possess any positive solution in* **R**. *In particular, if* $\overline{K}(r)$ *satisfies* (4.34) *and* $0 \leqslant \overline{K}(r) \leqslant C/r^{(\sigma+1)}$ *for* $r \geqslant R_1$ *for some constants* $C > 0$ *and* $R_1 > 0$, *then* $\overline{K}(r)$ *also satisfies* (4.35) *and hence* (1.1) *does not possess any positive solution in* **R**.

PROOF. Assume on the contrary that (1.1) has a positive solution $u(x)$ in **R**. Then as in the proof of Theorem 4.3, we have (4.24)–(4.27). Now the function $\tilde{K}(r) = r^{(\sigma-1)}\overline{K}(r)$ satisfies the assumptions of Theorem 2.3. Hence there is no positive function v satisfying (4.27). This contradiction proves the theorem. Q.E.D.

THEOREM 4.6. *Let* $K(x) \geqslant 0$ *be a continuous function in* **R** *and* $\tilde{K}(r)$ *be a continuous function in* $[0, \infty)$. *Let the average* $\overline{K}(r)$ *of* $K(x)$ *in the sense* (4.18) *satisfy*

$$\overline{K}(r) \geqslant 0 \quad \text{if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1},$$
$$\overline{K}(r) \geqslant \tilde{K}(r - \beta_i) \quad \text{if } \alpha_i + \beta_i \leqslant r \leqslant \alpha_{i+1} + \beta_i$$

for $i = 0, 1, 2, \ldots$, *where* $\{\alpha_i\}_{i=0}^{\infty}$ *is a strictly increasing sequence satisfying* $\alpha_0 = 0$ *and* $\lim_{n\to\infty} \alpha_n = \infty$, *and* $\{\beta_i\}_{i=0}^{\infty}$ *is a nondecreasing sequence satisfying* $\beta_0 = 0$ *and* $\beta_i/\alpha_i \leqslant M$ *for some* $M > 0$ *and for* $i \geqslant 1$. *If*

$$\begin{cases} u''(r) = \tilde{K}(r)u^{\sigma}(r) & \text{in } (0, \infty), \\ u(0) = \alpha > 0, \quad u'(0) = 0 \end{cases} \tag{4.36}$$

does not possess any positive solution in $[0, \infty)$ *for all* $\alpha > 0$, *then* (1.1) *does not possess any positive solution in* **R**.

PROOF. Assume that (1.1) has a positive solution $u(x)$ in **R**. Then we have as in the proof of Theorem 4.3,

$$\bar{u}(r) \geqslant \alpha + \int_0^r (r - t)\overline{K}(t)\bar{u}^{\sigma}(t)\, dt. \tag{4.37}$$

Let

$$v(r) = \bar{u}(r + \beta_i) \quad \text{if } \alpha_i \leqslant r < \alpha_{i+1} \tag{4.38}$$

for $i = 0, 1, 2, \ldots$. As in the proof of Theorem 2.4, we have

$$v(r) \geqslant \alpha + \int_0^r (r - t)\tilde{K}(t)v^{\sigma}(t)\, dt. \tag{4.39}$$

Now we can let X denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{y \in X : \alpha \leqslant y(r) \leqslant v(r) \text{ for } r \geqslant 0\}, \tag{4.40}$$

where v is defined in (4.38). Clearly, Y is a closed convex subset of X. We define the mapping T by

$$(Ty)(r) = \alpha + \int_0^r (r - t)\tilde{K}(t)y^{\sigma}(t)\, dt. \tag{4.41}$$

Then it is easy to verify that (i) $TY \subset Y$, (ii) T is continuous and (iii) TY is precompact. Hence T has a fixed point in Y. Thus (4.36) has a solution for this α. This contradiction completes the proof. Q.E.D.

PART II. $\Delta u = K(x)e^{2u}$

5. The case $n \geqslant 3$. In this case, the existence results are very similar to that of §2. Ni [**14**] proves that, if $|K(x)| \leqslant C/|x_1|^l$ for $|x_1|$ large and uniformly in x_2 for some $l > 2$, then equation (1.2) possesses infinitely many bounded solutions in $\mathbf{R}^n = \mathbf{R}^m \times \mathbf{R}^{n-m}$, where $x = (x_1, x_2)$ and $m \geqslant 3$. Later on, Kusano and Oharu [**7**] extend the result to the case where $|K(x)| \leqslant \tilde{K}(|x_1|)$ for all $x = (x_1, x_2) \in \mathbf{R}^m \times \mathbf{R}^{n-m}$ and $\int_0^\infty t\tilde{K}(t)\,dt < \infty$. On the other hand, when $K(x) \geqslant 0$ in (1.2), Oleinik [**15**] shows that if $K(x) \geqslant C/|x|^P$ at infinity for some $P < 2$, then (1.2) has no solution in $\mathbf{R}^n$. The case when $K(x)$ behaves like $C/|x|^2$ at infinity is left unsettled for $n \geqslant 3$. In this section, we give several theorems to settle the nonexistence question of (1.2), in particular we settle the case when $K(x)$ behaves like $C/|x|^2$ at infinity.

We need some notations first. Let u be a smooth function in $\mathbf{R}^n$ and $K(x) \geqslant 0$ be a continuous function in $\mathbf{R}^n$. Following Ni [**13**] and Sattinger [**16**], we define the averages of u and K by $\bar{u}(r)$ and $\overline{K}(r)$,

$$\bar{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x)\,dS, \tag{5.1}$$

$$\overline{K}(r) = \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right)^{-1}. \tag{5.2}$$

We have

LEMMA 5.1. *Let $u(x)$ be a solution of* (1.2) *in* $\mathbf{R}^n$ *and* $K(x) \geqslant 0$. *Then $\bar{u}(r)$ satisfies*

$$\begin{cases} \bar{u}''(r) + \dfrac{n-1}{r}\bar{u}'(r) \geqslant \overline{K}(r)e^{2\bar{u}(r)}, & r \in (0, \infty), \\ \bar{u}(0) = u(0), \qquad \bar{u}'(0) = 0. \end{cases} \tag{5.3}$$

PROOF. From the definition of $\bar{u}$, we have

$$\bar{u}'(r) = \frac{1}{\omega_n} \int_{|\xi|=1} \nabla u(r\xi) \cdot \xi\,dS = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \sum_i u_{x_i}\xi_i\,dS.$$

Thus,

$$\begin{aligned} &\omega_n\left(r^{n-1}\bar{u}'(r) - R^{n-1}\bar{u}'(R)\right) \\ &\quad = \int_D \Delta u\,dx = \int_R^r \left(\int_{|x|=t} \Delta u\,dS \right) dt \end{aligned} \tag{5.4}$$

where $D = \{x \in \mathbf{R}^n\colon R < |x| < r\}$. Hence we have

$$\omega_n\left(r^{n-1}\bar{u}'(r)\right)' = \int_{|x|=r} \Delta u\,dS = \int_{|x|=r} K(x)e^{2u(x)}\,dS. \tag{5.5}$$

Now Jensen's and Cauchy-Schwarz's inequalities give

$$\begin{aligned} e^{2\bar{u}(r)} = \left(e^{\bar{u}(r)}\right)^2 &\leqslant \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} e^{u(x)}\,dS \right)^2 \\ &\leqslant \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} K(x)e^{2u(x)}\,dS \right) \left(\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \frac{dS}{K(x)} \right). \end{aligned} \tag{5.6}$$

Hence

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} K(x) e^{2u(x)} dS \geqslant \overline{K}(r) e^{2\bar{u}(r)}. \tag{5.7}$$

Combining (5.5) and (5.7), we obtain the first equation of (5.3). $\bar{u}(0) = u(0)$ and $\bar{u}'(0) = 0$ can also be easily obtained. This completes the proof. Q.E.D.

Now we can state our main nonexistence theorems.

THEOREM 5.1. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^n$. If $\overline{K}(r)$, as defined in* (5.2), *satisfies*

$$\overline{K}(r) \geqslant C/r^2 \tag{5.8}$$

for $r \geqslant R_0$ for some constant $C > 0$, then equation (1.2) *does not possess any locally bounded solution in $\mathbf{R}^n$.*

PROOF. Assume that u is a locally bounded solution of (1.2) in $\mathbf{R}^n$. Then the average $\bar{u}$ satisfies (5.3) from Lemma 5.1. Let $\bar{u}(0) = u(0) = \alpha$. Then $\bar{u}$ also satisfies

$$\bar{u}'(r) \geqslant \int_0^r \left(\frac{s}{r}\right)^{n-1} \overline{K}(s) e^{2\bar{u}(s)} ds, \tag{5.9}$$

$$\bar{u}(r) \geqslant \alpha + \frac{1}{n-2} \int_0^r s\overline{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] e^{2\bar{u}(s)} ds. \tag{5.10}$$

Hence

$$\begin{aligned} \bar{u}(r) &\geqslant \alpha + \frac{1}{n-2} \int_0^{r/2} s\overline{K}(s) \left[1 - \left(\frac{1}{2}\right)^{n-2}\right] e^{2\alpha} ds \\ &= \alpha + \frac{1}{n-2} \cdot e^{2\alpha} \cdot \left[1 - \left(\frac{1}{2}\right)^{n-2}\right] \cdot \int_0^{r/2} s\overline{K}(s)\, ds. \end{aligned} \tag{5.11}$$

Thus there exists a constant R_0, such that $\bar{u}(R_0) \geqslant 1$. For $r \geqslant R_0$, we have

$$\begin{aligned} \bar{u}(r) &\geqslant 1 + \frac{1}{n-2} \int_{R_0}^r s\overline{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] e^{2\bar{u}(s)} ds \\ &\geqslant 1 + \frac{1}{n-2} \int_{R_0}^r s\overline{K}(s) \left[1 - \left(\frac{s}{r}\right)^{n-2}\right] \bar{u}^2(s)\, ds. \end{aligned} \tag{5.12}$$

In view of (5.8) and the proof of Theorem 2.1, we conclude that no function $\bar{u}$ can satisfy (5.12) in $[R_0, \infty)$. This completes the proof. Q.E.D.

THEOREM 5.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^n$. If $\overline{K}(r)$, as defined in* (5.2), *satisfies*

(5.13) *there exist $\alpha > 0$, $R_0 > 0$ and $C > 0$, such that*

$$\overline{K}(r) \geqslant C/r^\alpha \quad \text{for } r \geqslant R_0,$$

(5.14) *there exist $\varepsilon > 0$ and $P > 2$, such that*

$$\int_R^{(P-1)R} r\overline{K}(r)\, dr \geqslant \varepsilon \quad \text{for } R \geqslant R_0,$$

then equation (1.2) *does not possess any locally bounded solution in $\mathbf{R}^n$.*

PROOF. Assume that u is a locally bounded solution of (1.2) in $\mathbf{R}^n$. Then as in the proof of Theorem 5.1, we have (5.9)–(5.12). But from (5.13), (5.14) and Theorem 2.2, there is no function $\bar{u}(r)$ defined on $[R_0, \infty)$ satisfying (5.12). This contradiction proves the theorem. Q.E.D.

THEOREM 5.3. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function. If $\bar{K}(r)$, as defined in* (5.2), *satisfies*

$$\int_0^r s\bar{K}(s)\,ds \text{ is strictly increasing in } [0, \infty) \text{ and } \quad \int_0^\infty s\bar{K}(s)\,ds = \infty, \tag{5.15}$$

$$\left(\frac{s}{r}\right)^m \leqslant \int_0^s t\bar{K}(t)\,dt \Big/ \int_0^r t\bar{K}(t)\,dt \text{ for some } m > 0 \text{ and for all } r \geqslant s \geqslant R_0 > 0. \tag{5.16}$$

Then equation (1.2) *does not possess any locally bounded solution in* $\mathbf{R}^n$. *In particular, if $\bar{K}(r)$ satisfies* (5.15) *and $0 \leqslant \bar{K}(r) \leqslant C/r^2$ for $r \geqslant R_1$ for some constants $C > 0$ and $R_1 > 0$, then $\bar{K}(r)$ also satisfies* (5.16) *and hence* (1.2) *does not possess any locally bounded solution in* $\mathbf{R}^n$.

PROOF. Using the proofs of Theorems 5.1 and 2.3, we can easily obtain a proof. We omit the details. Q.E.D.

THEOREM 5.4. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^n$ and $\tilde{K}(t)$ be a locally Hölder continuous function on $[0, \infty)$. Let the average $\bar{K}(r)$ of $K(x)$ in the sense of* (5.2) *satisfy*

$$\bar{K}(r) \geqslant 0 \quad \text{if } \alpha_{i+1} + \beta_i < r < \alpha_{i+1} + \beta_{i+1},$$
$$\bar{K}(r) \geqslant \tilde{K}(r - \beta_i) \quad \text{if } \alpha_i + \beta_i \leqslant r \leqslant \alpha_{i+1} + \beta_i$$

for $i = 0, 1, 2, \ldots$, where $\{\alpha_i\}_{i=0}^\infty$ and $\{\beta_i\}_{i=0}^\infty$ are two sequences satisfying the same conditions as in Theorem 2.4. *If*

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) = \tilde{K}(r)e^{2u(r)} & \text{in } (0, \infty), \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{5.17}$$

does not possess any locally bounded solution in $[0, \infty)$ for any real number α, then (1.2) *does not possess any locally bounded solution in* $\mathbf{R}^n$.

PROOF. The proof is similar to that of Theorem 2.4. Hence we omit the details. Q.E.D.

6. The case $n = 2$. In the case $n = 2$ and $K(x) \geqslant 0$, Ni **[14]** shows that: If $K(x) \not\equiv 0$ and $K(x) \leqslant C/|x|^l$ at infinity for some $l > 2$, then for every $\alpha \in (0, \beta)$ where $\beta = \min\{8, (l-2)/3\}$, there exists a solution u of (1.2) such that

$$\log|x|^\alpha - C' \leqslant u(x) \leqslant \log|x|^\alpha + C''$$

for $|x|$ large, where C' and C'' are two constants.

Later, McOwen [**10, 11**] improves this result by giving a sharp bound on β and sharp behavior of u at infinity. For the nonexistence results, Sattinger [**16**] proves

Let K be a smooth function on $\mathbf{R}^2$. If $K \geqslant 0$ on $\mathbf{R}^2$ and $K(x) \geqslant C/|x|^2$ at infinity, then (1.2) has no solution on $\mathbf{R}^2$. Ni [**14**] improves Sattinger's result to include the K such as $K = (1 + \sin r)/r^2$.

In this section, we give an existence result which overlaps parts of the results of Ni [**14**] and McOwen [**10, 11**] but with different method. We also give some nonexistence results improving Ni's result.

THEOREM 6.1. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function on $\mathbf{R}^2$. Let $K_1(r)$ and $K_2(r)$ be two locally Hölder continuous functions on* $[0, \infty)$. *If*

$$K_1(0) > 0, \tag{6.1}$$

$$0 \leqslant K_1(|x|) \leqslant K(x) \leqslant K_2(|x|) \quad \text{for all } x \in \mathbf{R}^2, \tag{6.2}$$

$$\text{there exists } \alpha > 0 \text{ such that } \int_0^\infty s^{(1+2\alpha)} K_2(s)\, ds < \infty, \tag{6.3}$$

then (1.2) *has infinitely many solutions on $\mathbf{R}^2$ with logarithmic growth at infinity.*

PROOF. Consider the equations

$$\Delta v = K_1(|x|) e^{2v}, \qquad x \in \mathbf{R}^2, \tag{6.4}$$

$$\Delta w = K_2(|x|) e^{2w}, \qquad x \in \mathbf{R}^2. \tag{6.5}$$

From (6.2), it is easy to see that a solution v of (6.4) is a supersolution of (1.2) and a solution w of (6.5) is a subsolution of (1.2) in $\mathbf{R}^2$. It is natural to seek solutions of v and w depending only on $|x|$. Considser now (6.5). We try to find a solution $w(|x|)$ of (6.5) with $w(0) = \beta$ and $w'(0) = 0$. Then (6.5) is equivalent to the following integral equation

$$w(r) = \beta + \int_0^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)}\, ds. \tag{6.6}$$

Now we choose $0 < \alpha' < \alpha$ and β such that

$$\int_0^e s \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)}\, ds < \frac{1}{2}, \tag{6.7}$$

$$\int_0^e s K_2(s) e^{2(\beta+1)}\, ds < \frac{\alpha'}{2}, \tag{6.8}$$

$$\int_e^\infty s^{(1+2\alpha')} K_2(s) e^{2(\beta+1)}\, ds < \frac{\alpha'}{2}, \tag{6.9}$$

$$\int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)}\, ds < \frac{1}{2}. \tag{6.10}$$

Define the function $A_\beta(r)$ by

$$A_\beta(r) = (\beta + 1) \quad \text{if } 0 \leqslant r \leqslant e, \tag{6.11}$$

$$A_\beta(r) = (\beta + 1) + \alpha' \log(r/e) \quad \text{if } e \leqslant r.$$

Now let X denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{w \in X\colon \beta \leqslant w(r) \leqslant A_\beta(r),\ r \in [0, \infty)\}. \tag{6.12}$$

It is easy to see that Y is a closed convex subset of X. Let T be the mapping

$$(Tw)(r) = \beta + \int_0^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)}\, ds. \tag{6.13}$$

We shall prove that T is a continuous mapping from Y into itself such that TY is relatively compact.

First, we verify that $TY \subset Y$. Assume $w \in Y$. Hence we have

$$\beta \leqslant w(r) \leqslant A_\beta(r) \quad \text{for } r \in [0, \infty). \tag{6.14}$$

It is easy to see that Tw is also continuous and $\beta \leqslant Tw(r)$ for $r \in [0, \infty)$. Now for $0 \leqslant r \leqslant e$, we have

$$\begin{aligned}(Tw)(r) &= \beta + \int_0^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)}\, ds \\ &\leqslant \beta + \int_0^e s \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)}\, ds \\ &\leqslant (\beta + 1) = A_\beta(r).\end{aligned} \tag{6.15}$$

For $e \leqslant r$, we have

$$\begin{aligned}(Tw)(r) &= \beta + \int_0^e s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)}\, ds \\ &\quad + \int_e^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2w(s)}\, ds \\ &\leqslant \beta + \int_0^e s \log\left(\frac{r}{s}\right) K_2(s) e^{2A_\beta(s)}\, ds \\ &\quad + \int_e^r s \log\left(\frac{r}{s}\right) K_2(s) e^{2A_\beta(s)}\, ds \\ &\leqslant \beta + \log\left(\frac{r}{e}\right) \int_0^e s K_2(s) e^{2(\beta+1)}\, ds \\ &\quad + \int_0^e s \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)}\, ds \\ &\quad + \log\left(\frac{r}{e}\right) \int_e^\infty s^{(1+2\alpha')} K_2(s) e^{2(\beta+1)}\, ds \\ &\quad + \int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) K_2(s) e^{2(\beta+1)}\, ds \\ &\leqslant \beta + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} \\ &= (\beta + 1) + \alpha' \log\left(\frac{r}{e}\right) \\ &= A_\beta(r).\end{aligned} \tag{6.16}$$

This verifies that $TY \subset Y$.

Now let $\{w_m\}_{m=1}^{\infty} \subset Y$ be a sequence converges to $w \in Y$ in the space X. Then $\{w_m\}$ converges to w uniformly on any compact interval on $[0, \infty)$. Now

$$|Tw_m(r) - Tw(r)| \leqslant \int_0^r s\log\left(\frac{r}{s}\right) K_2(s)|e^{2w_m(s)} - e^{2w(s)}|\, ds \tag{6.17}$$

But

$$\begin{aligned} s\log\left(\frac{r}{s}\right) K_2(s)|e^{2w_m(s)} - e^{2w(s)}| &\leqslant s\log\left(\frac{r}{s}\right) K_2(s)(e^{2A_\beta(s)} - e^{2\beta}) \\ &\leqslant s\log\left(\frac{r}{s}\right) K_2(s) e^{2A_\beta(s)} \end{aligned} \tag{6.18}$$

and $s\log(r/s)K_2(s)e^{2A_\beta(s)}$ is integrable. Hence from (6.17) and the uniform convergence of w_m to w on any compact interval, we conclude that Tw_m converges to Tw in X. This verifies that T is continuous in Y. We can easily compute that

$$\begin{aligned} (Tw)'(r) &= \int_0^r \left(\frac{s}{r}\right) K_2(s) e^{2w(s)}\, ds \\ &\leqslant \int_0^r \left(\frac{s}{r}\right) K_2(s) e^{2A_\beta(s)}\, ds. \end{aligned} \tag{6.19}$$

Hence, on any compact interval of $[0, \infty)$, TY is uniformly bounded and equicontinuous. This proves that TY is relatively compact in Y. Thus we can apply the Schauder-Tychonoff fixed point theorem to conclude that T has a fixed point w in Y. This fixed point w is a solution of (6.6) and hence a solution of (6.5). Note that, when we have a solution w of (6.6) with a given β, then we also have a solution w of (6.6) with β replaced by smaller β's.

Similarly, we can construct solution $v(|x|)$ of (6.4) such that $v(0) = \beta'$ and $v'(0) = 0$. For a given β', since $K_1(0) > 0$, we can choose $\beta < \beta'$, such that (6.6) has a solution w and $w(r) \leqslant v(r)$ for all $r \in [0, \infty)$. Using Theorem 2.10 of Ni **[13]**, we conclude that (1.2) has a solution $u(x)$ between $w(|x|)$ and $v(|x|)$. Now we can choose another β' smaller than this β to repeat the arguments. This completes the proof of this theorem. Q.E.D.

THEOREM 6.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$. If $\overline{K}(r)$, as defined in* (5.2), *satisfies*

$$\overline{K}(r) \geqslant C/r^2(\log r)^a \tag{6.20}$$

for $r \geqslant R_0$ for some constants $C > 0$ and $a > 0$, then equation (1.2) *does not possess any locally bounded solution in $\mathbf{R}^2$.*

PROOF. Assume that u is a locally bounded solution of (1.2) in $\mathbf{R}^2$. Then the average $\bar{u}$ satisfies (5.3) for $n = 2$. Letting $\bar{u}(0) = \beta = u(0)$, we have

$$\bar{u}'(r) \geqslant \int_0^r \left(\frac{s}{r}\right) \overline{K}(s) e^{2\bar{u}(s)}\, ds, \tag{6.21}$$

$$\bar{u}(r) \geqslant \beta + \int_0^r s\log\left(\frac{r}{s}\right) \overline{K}(s) e^{2\bar{u}(s)}\, ds. \tag{6.22}$$

Without loss of generality, we may assume that $K(0) > 0$ and hence $\bar{K}(0) > 0$. For $r \geqslant e$, we have

$$\begin{aligned}(6.23)\qquad \bar{u}(r) &\geqslant \beta + \int_0^1 s \log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds \\ &\quad + \int_1^r s\log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds \\ &\geqslant \beta + \int_0^1 s\log r\bar{K}(s)e^{2\beta}\,ds + \int_1^r s\log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds \\ &\geqslant \beta + C_1\log r + \int_e^r s\log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds.\end{aligned}$$

Thus there exists a constant R_0 such that, for $r \geqslant R_0$,

$$\begin{aligned}(6.24)\qquad \bar{u}(r) &\geqslant C_2\log r + \int_e^r s\log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds \\ &\geqslant C_2\log r + \int_{R_0}^r s\log\left(\frac{r}{s}\right)\bar{K}(s)e^{2\bar{u}(s)}\,ds\end{aligned}$$

for some $C_2 > 0$. Let

$$(6.25)\qquad \bar{u}(r) = \tfrac{1}{2}C_2\log r + v(r) \quad \text{for } r \geqslant R_0.$$

From (6.24), we have

$$\begin{aligned}(6.26)\qquad v(r) &\geqslant \frac{1}{2}C_2\log r + \int_{R_0}^r s\log\left(\frac{r}{s}\right)\bar{K}(s)s^{C_2}e^{2v(s)}\,ds \\ &\geqslant \frac{1}{2}C_2\log r + \int_{R_0}^r s\log\left(\frac{r}{s}\right)\bar{K}(s)s^{C_2}v^2(s)\,ds.\end{aligned}$$

But from assumption (6.20), we have

$$(6.23)\qquad \bar{K}(s)s^{C_2} \geqslant C/s^{2-C_2}(\log s)^a \geqslant C/s^2$$

for $s \geqslant R_1 > R_0$. Hence from Theorem 3.1, there is no v in $[R_0, \infty)$ satisfying (6.26). This completes the proof of this theorem.

THEOREM 6.3. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$. If $\bar{K}(r)$, as defined in* (5.2), *satisfies*

(6.24) $\int_0^r s^{1+\alpha}\bar{K}(s)\,ds$ *is monotonically strictly increasing in* $[0, \infty)$ *for all* $\alpha > 0$.

(6.25) *For given any $\alpha > 0$, there exists an $R_\alpha > 0$ such that*

$$\left(\frac{\log s}{\log r}\right)^m \leqslant \int_0^s t^{1+\alpha}\bar{K}(t)\,dt \Big/ \int_0^r t^{1+\alpha}\bar{K}(t)\,dt$$

for some $m > 0$ and for all $r \geqslant s \geqslant R_\alpha$, then (1.2) *does not possess any locally bounded solution in* $\mathbf{R}^2$.

PROOF. Assume that u is a locally bounded solution of (1.2) in $\mathbf{R}^2$. Then as in the proof of Theorem 6.2, we have (6.21)–(6.26). Now we can let $w(r)\log r = v(r)$ for $r \geqslant R_0$. Then from (6.26), we have

$$w(r) \geqslant \tfrac{1}{2}C_2 + \int_{R_0}^{r} s\left(1 - \frac{\log s}{\log r}\right)\overline{K}(s)s^{C_2}v^2(s)\,ds. \tag{6.27}$$

Now using a similar argument as in the proof of Theorem 3.3, we conclude that there is no function w satisfying (6.27). This contradiction proves the theorem. Q.E.D.

THEOREM 6.4. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^2$ and $\tilde{K}(t)$ be a locally Hölder continuous function on $[0, \infty)$. Let the average $\overline{K}(r)$ of $K(x)$ in the sense of (5.2) satisfy the same assumptions as in Theorem 5.4. If*

$$\begin{cases} u''(r) + \dfrac{u'(r)}{r} = \tilde{K}(r)e^{2u(r)} & \text{in } (0, \infty), \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{6.28}$$

does not possess any locally bounded solution in $[0, \infty)$ for any real number α, then (1.2) does not possess any locally bounded solution in $\mathbf{R}^2$.

PROOF. The proof is similar to that of Theorem 2.4. Hence we omit the details. Q.E.D.

7. The case $n = 1$. In this case, we consider only the situation $K(x) \geqslant 0$ in (1.2). We give a main existence result which has an extension to the higher-dimensional case. We also give some nonexistence results.

THEOREM 7.1. *Let $K(x) \geqslant 0$ be a Hölder continuous function in $\mathbf{R}$. If $K(0) > 0$ and there exists an $\alpha > 0$, such that*

$$\int_{-\infty}^{\infty} e^{2\alpha|x|}K(x)\,dx < \infty, \tag{7.1}$$

then (1.2) has infinitely many locally bounded solutions in $\mathbf{R}$ with linear growth at $|x| = \infty$.

PROOF. We shall seek solution u such that $u(0) = \beta$ and $u'(0) = 0$. Consider now $x \geqslant 0$. In this situation, (1.2) is equivalent to the integral equation

$$u(x) = \beta + \int_0^x (x - t)K(t)e^{2u(t)}\,dt, \qquad x \geqslant 0. \tag{7.2}$$

Now choose $\beta \in \mathbf{R}$ so that

$$\int_0^1 K(t)e^{2(\beta+1)}\,dt \leqslant \min\left\{\frac{\alpha}{2}, 1\right\}, \tag{7.3}$$

$$\int_1^{\infty} K(t)e^{2\alpha t}e^{2(\beta+1)}\,dt \leqslant \frac{\alpha}{2}. \tag{7.4}$$

Let

$$A(x) = \begin{cases} (\beta + 1) & \text{if } 0 \leqslant x \leqslant 1, \\ (\beta + 1) + \alpha x & \text{if } 1 < x. \end{cases}$$

As in the proofs of Theorems 2.4 and 3.4, we let X denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology and consider the set

$$Y = \{ y \in X\colon \beta \leqslant y(x) \leqslant A(x) \text{ for } x \geqslant 0 \}.$$

Clearly, Y is a closed convex subset of X. Now define the mapping T by

$$(7.5)\qquad (Ty)(x) = \beta + \int_0^x (x-t)K(t)e^{2y(t)}\,dt.$$

If $y \in Y$, then $\beta \leqslant y(x) \leqslant A(x)$. Hence we have

$$(7.6)\qquad (Ty)(x) = \beta + \int_0^x (x-t)K(t)e^{2y(t)}\,dt \geqslant \beta.$$

On the other hand, for $0 \leqslant x \leqslant 1$, we have

$$(7.7)\qquad \begin{aligned}(Ty)(x) &= \beta + \int_0^x (x-t)K(t)e^{2y(t)}\,dt \\ &\leqslant \beta + \int_0^1 K(t)e^{2(\beta+1)}\,dt \\ &\leqslant \beta + 1 = A(x).\end{aligned}$$

For $1 < x$, we have

$$(7.8)\qquad \begin{aligned}(Ty)(x) &= \beta + \int_0^1 (x-t)K(t)e^{2y(t)}\,dt + \int_1^x (x-t)K(t)e^{2y(t)}\,dt \\ &\leqslant \beta + x\cdot\int_0^1 K(t)e^{2(\beta+1)}\,dt + x\cdot\int_1^\infty K(t)e^{2\alpha t}e^{2(\beta+1)}\,dt \\ &\leqslant \beta + \frac{\alpha}{2}\cdot x + \frac{\alpha}{2}x \leqslant (\beta+1) + \alpha x = A(x).\end{aligned}$$

Hence T maps Y into itself. As in the proofs of Theorems 2.4, 3.4 and 4.1, we can easily verify that T is continuous and TY is precompact. Hence T has a fixed point $y \in Y$. This fixed point y is a solution of (1.2) for $x \geqslant 0$ with $y(0) = \beta$ and $y'(0) = 0$.

Similarly, we can find a solution of (1.2) for $x \leqslant 0$ with $y(0) = \beta$ and $y'(0) = 0$ provided that $\beta \in \mathbf{R}$ is properly selected. It is also easy to see that if y is a solution of (1.2) with $y(0) = \beta$ and $y'(0) = 0$, then there is also solution y with $y(0) = \beta'$ and $y'(0) = 0$ provided that $\beta' < \beta$. The linear growth of solutions at $|x| = \infty$ can be easily established as in the proof of Theorem 4.1. This completes the proof of this theorem. Q.E.D.

We can apply this theorem to the higher-dimensional case as used in Ni [**13, 14**] and Kawano, Kusano and Naito [**3**].

THEOREM 7.2. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$. Let $\phi_*(x_1)$ and $\phi^*(x_1)$ be two locally Hölder continuous function in $\mathbf{R}$. If*

$$(7.9)\qquad 0 \leqslant \phi_*(x_1) \leqslant K(x) \leqslant \phi^*(x_1) \quad \textit{for all } x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1},$$

$$(7.10)\qquad \phi_*(0) > 0 \textit{ and } \int_{-\infty}^{\infty} e^{2\alpha|x_1|}\phi^*(x_1)\,dx_1 < \infty \quad \textit{for some } \alpha > 0,$$

then equation (1.2) *has infinitely many locally bounded solutions in* $\mathbf{R}^n$.

PROOF. The proof is actually similar to that of Theorem 4.2. We omit the details. Q.E.D.

Now let u be smooth function on $\mathbf{R}$ and $K(x) \geqslant 0$ be a continuous function on $\mathbf{R}$. We define the averages $\bar{u}$ and $\overline{K}$ by

$$\bar{u}(r) = \tfrac{1}{2}[u(r) + u(-r)], \qquad r \geqslant 0, \tag{7.11}$$

$$\overline{K}(r) = \left[\tfrac{1}{2}\left(K(r)^{-1} + K(-r)^{-1}\right)\right]^{-1}, \qquad r \geqslant 0. \tag{7.12}$$

Our nonexistence results are

THEOREM 7.3. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function on* $\mathbf{R}$. *If the average $\overline{K}(r)$ of $K(x)$ in the sense of* (7.12) *satisfies*

$$\overline{K}(r) \geqslant C/r^{a} \tag{7.13}$$

for $r \geqslant R_0$ and for some constants $C > 0$, $a > 0$, then equation (1.2) *does not possess any locally bounded solution on* $\mathbf{R}$.

PROOF. Assume that $u(x)$ be a solution of (1.2) in $\mathbf{R}$. Then we have

$$\begin{aligned} \bar{u}''(r) &= \tfrac{1}{2}[u''(r) + u''(-r)] \\ &= \tfrac{1}{2}\left[K(r)e^{2u(r)} + K(-r)e^{2u(-r)}\right]. \end{aligned} \tag{7.14}$$

But we have

$$\begin{aligned} e^{2\bar{u}(r)} = \left(e^{\bar{u}(r)}\right)^2 &\leqslant \left[\tfrac{1}{2}\left(e^{u(r)} + e^{u(-r)}\right)\right]^2 \\ &\leqslant \left[\tfrac{1}{2}\left(K(r)e^{2u(r)} + K(-r)e^{2u(r)}\right)\right] \\ &\quad \cdot \left[\tfrac{1}{2}\left(K(r)^{-1} + K(-r)^{-1}\right)\right]. \end{aligned} \tag{7.15}$$

Hence we have

$$\bar{u}''(r) \geqslant \overline{K}(r)e^{2\bar{u}(r)}, \qquad r \geqslant 0. \tag{7.16}$$

It is also easy to see that $\bar{u}(0) = u(0)$ and $\bar{u}'(0) = 0$. From (7.16), we have

$$\bar{u}'(r) \geqslant \int_0^r \overline{K}(t)e^{2\bar{u}(t)}\,dt, \tag{7.17}$$

$$\bar{u}(r) \geqslant \beta + \int_0^r (r-t)\overline{K}(t)e^{2\bar{u}(t)}\,dt. \tag{7.18}$$

Without loss of generality, we may assume that $K(0) > 0$ and hence $\overline{K}(0) > 0$. For $r \geqslant 1$, we have

$$\begin{aligned} \bar{u}(r) &\geqslant \beta + \int_0^1 (r-t)\overline{K}(t)e^{2\bar{u}(t)}\,dt + \int_1^r (r-t)\overline{K}(t)e^{2\bar{u}(t)}\,dt \\ &\geqslant \beta + r\int_0^1 (1-t)\overline{K}(t)e^{2\beta}\,dt + \int_1^r (r-t)\overline{K}(t)e^{2\bar{u}(t)}\,dt \\ &\geqslant 2C_1 \cdot r + \int_{R_1}^r (r-t)\overline{K}(t)e^{2\bar{u}(t)}\,dt \end{aligned} \tag{7.19}$$

for $r \geqslant R_1 > 1$ and for some $C_1 > 0$. Now let $v(r) = \bar{u}(r) + C_1 \cdot r$. We have from (7.19)

$$v(r) \geqslant C_1 \cdot r + \int_{R_1}^{r} (r-t)\overline{K}(t)e^{2C_1 t} \cdot e^{2v(t)}\, dt. \tag{7.20}$$

Let $v(r) = w(r) \cdot r$, we have

$$w(r) \geqslant C_1 + \int_{R_1}^{r} \left(1 - \frac{t}{r}\right)\overline{K}(t)e^{2C_1 t} \cdot e^{2w(t)}\, dt. \tag{7.21}$$

Now let $\tilde{K}(t) = t^{-1}\overline{K}(t)e^{2C_1 t}$. We have from (7.13)

$$\tilde{K}(t) \geqslant C/t^2 \tag{7.22}$$

for $t \geqslant R_2 > R_1$ for some $C > 0$. But (7.21) becomes

$$w(r) \geqslant C_1 + \int_{R_1}^{r} t\left(1 - \frac{t}{r}\right)\tilde{K}(t)w^2(t)\, dt. \tag{7.23}$$

From Theorem 2.1, there is no function w satisfying (7.23). This contradiction proves the theorem. Q.E.D.

THEOREM 7.4. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function on* **R**. *If the average $\overline{K}(r)$ of $K(x)$ in the sense of* (7.12) *satisfies*

(7.24) $\int_0^r e^{\alpha s}\overline{K}(s)\, ds$ *is strictly increasing and* $\int_0^\infty e^{\alpha s}\overline{K}(s)\, ds = \infty$ *for all* $\alpha > 0$.

For any given $\alpha > 0$, there exists $R_\alpha > 0$, such that

$$\left(\frac{s}{r}\right)^m \leqslant \int_0^s e^{\alpha t}\overline{K}(t)\, dt \Big/ \int_0^r e^{\alpha t}\overline{K}(t)\, dt \tag{7.25}$$

for some $m > 0$ and for $r \geqslant s \geqslant R_\alpha$, then equation (1.2) *does not possess any locally bounded solution in* **R**.

PROOF. Using the proofs of Theorems 7.3 and 2.3, we can easily prove this theorem. We omit the details. Q.E.D.

THEOREM 7.5. *Let $K(x) \geqslant 0$ be a locally Hölder continuous function in* **R** *and $\tilde{K}(t)$ be a locally Hölder continuuos function in* $[0, \infty)$. *Let the average $\overline{K}(r)$ of $K(x)$ in the sense of* (7.12) *satisfy the same assumptions as in Theorem* 5.4. *If*

$$\begin{cases} u''(r) = \tilde{K}(r)e^{2u(r)} & \text{in } (0,\infty), \\ u(0) = \beta, \quad u'(0) = 0 \end{cases} \tag{7.26}$$

does not possess any locally bounded solution in $[0, \infty)$ *for any real number β, then equation* (1.2) *does not possess any locally bounded solution in* **R**.

PROOF. The proof is quite similar to that of Theorem 2.4. Hence we omit it. Q.E.D.

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